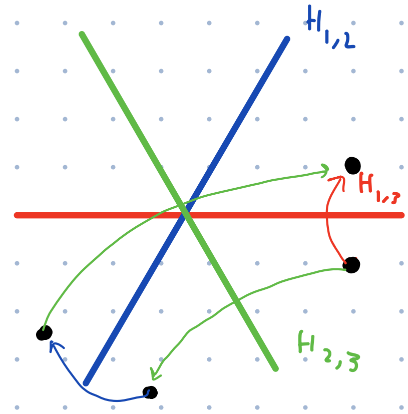
2025/11/07

The Affine Symmetric Group

Defn: $V := \{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 0 \}$



$n=3, \quad V \simeq \mathbb{R}^2$

$t_{1,3} = s_2 s_1 s_2$

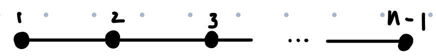
For $1 \le i < j \le n$, consider $H_{i,j} = \{ x \in V \mid x_i - x_j = 0 \}$

$t_{i,j}$ = reflection fixing $H_{i,j}$

$\mathfrak{S}_n \cong \langle t_{i,j} \mid 1 \le i < j \le n \rangle = \langle \overset{s_i}{t_{i,i+1}} \mid 1 \le i \le n-1 \rangle$

$\mathfrak{S}_n$ is also a Coxeter group

$$= \left\langle s_1, \dots, s_{n-1} \;\middle|\; \begin{array}{l} s_i^2 = e, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \\ s_i s_j = s_j s_i \text{ if } |i-j| \ge 2 \end{array} \right\rangle$$

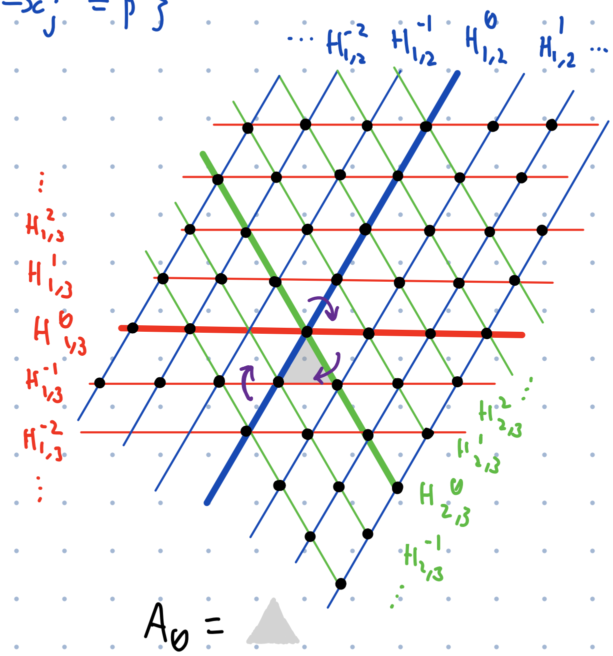


Dynkin diagram for $\mathfrak{S}_n$

We now define the Affine Symmetric Group analogously:

For $1 \le i < j \le n$, $p \in \mathbb{Z}$ consider $H_{i,j}^p = \{ x \in V \mid x_i - x_j = p \}$

$t_{i,j}^p$ = reflection fixing $H_{i,j}^p$



Defn: $\tilde{\mathfrak{S}}_n := \langle t_{i,j}^p \mid 1 \le i, j \le n, \; p \in \mathbb{Z} \rangle$

Defn: An alcove is a connected component of $V \setminus \bigcup_{i,j,p} H_{i,j}^p$.

Fix one to be A_0.

$\tilde{\mathfrak{S}}_n$ acts transitively & faithfully on the set of alcoves by permuting them

$\Rightarrow$ set of alcoves $\longleftrightarrow \tilde{\mathfrak{S}}_n$

$\tilde{\mathfrak{S}}_n = \langle \overset{s_n}{t_{(1,n)}^{-1}}, \overset{s_i}{t_{i,i+1}^{0}} \mid 1 \le i \le n-1 \rangle$

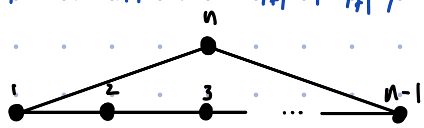
These gens are marked w/ $\curvearrowright$

The root lattice of V is the set of integer points. (intersection points • of lines)

One can say $\tilde{\mathfrak{S}}_n$ is the grp of rigid transformations preserving the root lattice.

Prop: $\tilde{\mathfrak{S}}_n$ is a Coxeter group:

$$\tilde{\mathfrak{S}}_n = \langle s_1,\dots,s_n \mid s_i^2 = e,\ \overbrace{s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\quad s_i s_j = s_j s_i \ \text{ for } |i-j| \geq 2}^{\text{indexing modulo } n} \rangle$$



Dynkin diagram for $\tilde{\mathfrak{S}}_n$

and a group of permutations:

$$\tilde{\mathfrak{S}}_n = \{ \text{bijections } u : \mathbb{Z} \to \mathbb{Z} \mid u(i+n) = u(i) + n,\ \sum_{i=1}^{n} u_i = \binom{n+1}{2} \}$$

$$\cup\!|$$

$$\mathfrak{S}_n = \mathrm{Stab}([n]) = \{ u \in \tilde{\mathfrak{S}}_n \mid \{u(1),\dots,u(n)\} = [n] \}$$

one can view $\tilde{\mathfrak{S}}_n$ as $\mathrm{Sym}(\mathbb{Z}) / \{ s_i \sim s_{i+n} \mid i \in \mathbb{Z} \}$

We can write elements of $\tilde{\mathfrak{S}}_n$ as $u = [a_1,\dots,a_n]$, $u(i) = a_i$

$s_i = [1, 2, \dots, i+1, i, \dots, n]$ $1 \leq i \leq n-1$, $s_n = [0, 2, \dots, n-1, n+1]$

With this notation

$$[a_1 \dots a_n]\, s_i = [a_1, \dots, a_{i+1}, a_i, \dots, a_n]$$
$$[a_1, \dots, a_n]\, s_n = [a_n - n, a_2, \dots, a_{n-1}, a_1 + n]$$

Defn: We may write any $u \in \tilde{\mathfrak{S}}_n$ as $s_{i_1} \cdots s_{i_\ell}$.
Let $\ell(u)$ be the minimum such ℓ.
Same defn as $\mathfrak{S}_n$, or any Coxeter group

Prop: $u \in \tilde{\mathfrak{S}}_n$, $u = [a_1, \dots, a_n]$

$$\ell(u) = \#\{ (i,j) \in [n] \times [n] \mid i < j,\ a_i > a_j \} + \sum_{1 \leq i < j \leq n} \left\lfloor \frac{|a_j - a_i|}{n} \right\rfloor$$
$$= \#\{ (i,j) \in [n] \times \mathbb{Z}_{>0} \mid i < j,\ a_i > a_j \}$$

For $u \in \mathfrak{S}_n$, $\ell(u) = \#\{ (i,j) \in [n] \times [n] \mid i < j,\ a_i > a_j \}$

Rmk: With the bijection $\tilde{\mathfrak{S}}_n \longleftrightarrow$ alcoves, the length $\ell(u)$ may be computed as the number of hyperplanes $H_{i,j}^{p}$ crossed along a shortest path from the alcove corresponding to e to the alcove corresponding to u.

Recall: $T = \{t_{i,j} \mid 1 \le i < j \le n\}$ the transpositions of $\mathfrak{S}_n$

For $u, v \in \mathfrak{S}_n$, say $u \xrightarrow{(t)} v$ if $\exists t \in T$, $v = ut$, and $\ell(v) = \ell(u) + 1$
Equiv, $\exists$ $1 \le i < j \le n$, $u(i) < u(j)$, $v = ut_{i,j}$

Bruhat order is given by $u \le v$ if $\exists$ $u_0, \dots, u_k$ $u = u_0 \to u_1 \to \cdots \to u_k = v$

Prop: Bruhat order can be computed via the diagram of $u \in \mathfrak{S}_n$

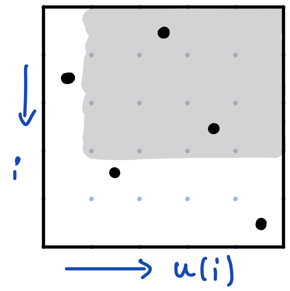


diagram for $u = 31425$
$r_u(3,2) = 2$

Defn: for $i,j \in [n]$,
$r_u(i,j) := \# \{k \le i \mid w(k) \le j\}$ is the rank function for

Then $u \le v \iff \forall$ $1 \le i, j \le n$, $r_u(i,j) \le r_v(i,j)$

Defn: For $i \not\equiv j \pmod n$, say $t_{i,j} := \prod_{k \in \mathbb{Z}} (i + kn, j + kn)$

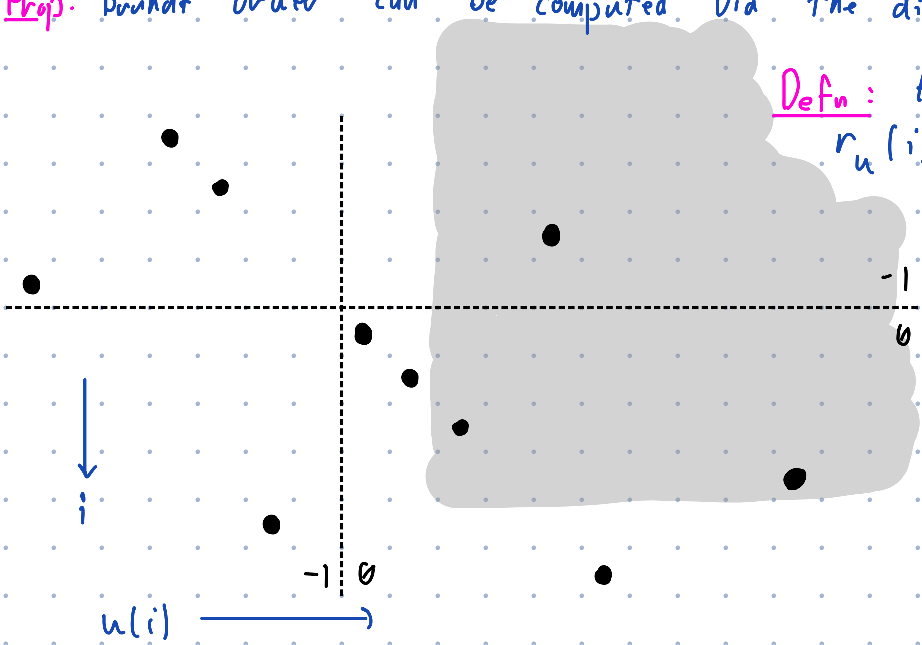
$\neq [a_1 = 1, \dots, a_i = j, \dots, a_j = i, \dots, a_n = n]$

$\widehat{T} := \{ t_{i, j+kn} \mid 1 \le i < j \le n, k \in \mathbb{Z}\}$

For $u, v \in \widetilde{\mathfrak{S}}_n$, say $u \xrightarrow{(t)} v$ if $\exists t \in \widetilde{T}$, $v = ut$, and $\ell(v) = \ell(u) + 1$
Equiv, $\exists$ $i, j \in \mathbb{Z}$, $i < j$, $i \not\equiv j \pmod n$, $u(i) < u(j)$, $v = ut_{i,j}$

Bruhat order is given by $u \le v$ if $\exists$ $u_0, \dots, u_k$ $u = u_0 \to u_1 \to \cdots \to u_k = v$

Prop: Bruhat order can be computed via the diagram of $u \in \widetilde{\mathfrak{S}}_n$



Defn: for $i,j \in \mathbb{Z}$
$r_u(i,j) := \# \{k \le i \mid w(k) \le j\}$

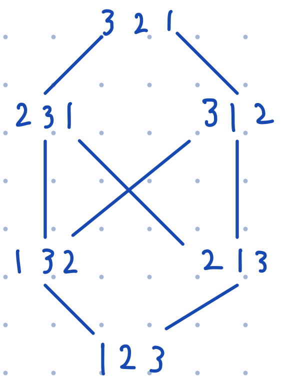
Then $u \le v$
$\iff \forall$ $i,j \in \mathbb{Z}$, $r_u(i,j) \le r_v(i,j)$

$u = [1\ 2\ 9\ -2\ 5]$

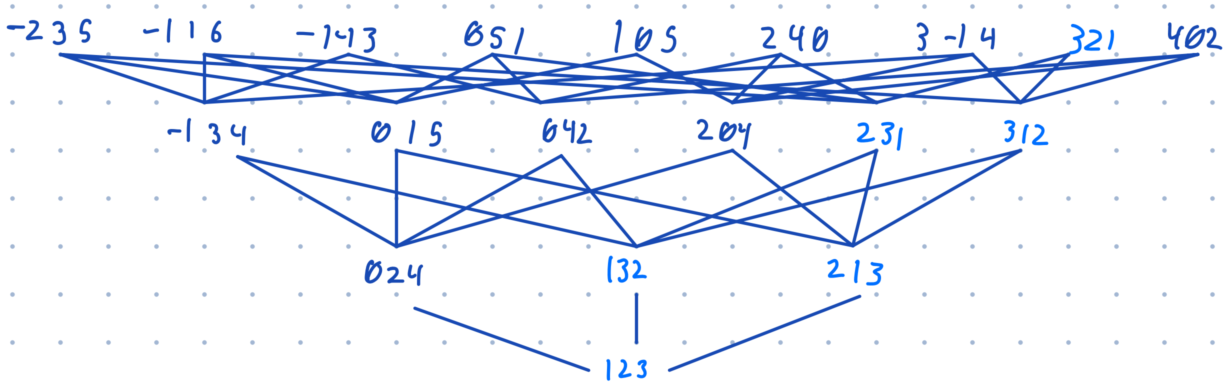
$r_u(3,2) = 3$

Rmk: $G_n \subseteq \tilde{G}_n$, and $u \leq v$ in $G_n \Leftrightarrow u \leq v$ in $\tilde{G}_n$

Ex: We can compute the Hasse diagrams for the Bruhat order



Bruhat order for G_3



Bruhat for $\tilde{G}_3$ for elements of rank ≤ 3

G_3 highlighted

2025/11/14

Defn: Parabolic Subgroups are subgroups of Coxeter groups generated by subsets of the Coxeter generators

Defn: $G_n = \langle s_1, \dots, s_{n-1} \rangle = \langle S \rangle$

$$(G_n)_{\hat{k}} = \langle s_1, \dots, s_{k-1}, s_{k+1}, \dots, s_{n-1} \rangle \quad \text{maximal parabolic}$$
$$= \mathrm{Stab}([k]) \simeq G_k \times G_{n-k}$$

Let $G_n^{(k)} := \{ w \in G_n \mid w(1) < \cdots < w(k),\ w(k+1) < \cdots < w(n) \}$ unique descent at k

Then every element $w \in G_n$ has a unique decomposition

$$w = u \cdot v,$$
$$u \in G_n^{(k)}, \quad v \in (G_n)_{\hat{k}}$$
$$\ell(w) = \ell(u) + \ell(v)$$

So $G_n^{(k)}$ gives coset reps of $G_n / (G_n)_{\hat{k}}$

Furthermore, $G_n^{(k)}$ is the set of coset reps of minimal length.

The story for $\widetilde{\mathfrak{S}}_n$ is the same.

Defn: $\widetilde{\mathfrak{S}}_n = \langle s_1, \ldots, s_n \rangle$

$$(\widetilde{\mathfrak{S}}_n)_{\hat{k}} = \langle s_1, \ldots, s_{k-1}, s_{k+1}, \ldots s_n \rangle = \mathrm{Stab}([k+1, \ldots, n+k])$$

$$(\widetilde{\mathfrak{S}}_n)_{\hat{n}} = \mathfrak{S}_n$$

$$\widetilde{\mathfrak{S}}_n^{(k)} = \{ w \in \widetilde{\mathfrak{S}}_n \mid w(1) < \cdots < w(k), \ w(k+1) < \cdots < w(n+1) \}$$

We enjoy the same decomposition and coset representation.

Ex: We consider $\widetilde{\mathfrak{S}}_n / \mathfrak{S}_n$
To find the coset rep of $w\mathfrak{S}_n$, we want the element of minimal length. This unique element u such that

$$\ell(u s_i) < \ell(u) \iff i = n$$

i.e. in window notation, $u = [u_1, \ldots, u_n]$ satisfies $u_1 < \cdots < u_n$ where $\{u_1, \ldots, u_n\} = \{w(1), \ldots, w(n)\}$

Ex: As another more concrete example, consider $n=5$ and $\widetilde{\mathfrak{S}}_5 / (\widetilde{\mathfrak{S}}_5)_{\hat{3}}$

We compute a coset representative of $w(\mathfrak{S}_5)_{\hat{3}}$

Let $w = [-3, 6, 3, -5, 14]$

here $k=3$

The rule is to write the entries $\{w(\ell n+k+1), \ldots, w(\ell n+k+n)\}$ in inc. order

$$\ell = 0 \qquad \{ \underset{-5}{w(4)}, \underset{14}{w(5)}, \underset{2}{w(6)}, \underset{11}{w(7)}, \underset{8}{w(8)} \}$$

$$\begin{matrix} -5 & & 2 & & 8 & & 11 & & 14 \\ [\,3 & < & 6 & < & 9 & & -5 & < & 2\,] \end{matrix}$$

rearrange

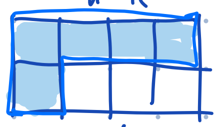
subtract 5 to get $w(1), \ldots, w(5)$

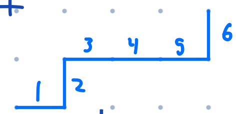
Extras from Dave

$G_n / G_k \times G_{n-k}$ has the k-Grassmannian permutations $\left\{ \begin{matrix} w(1) < \cdots < w(k) \\ w(k+1) < \cdots < w(n) \end{matrix} \right\}$

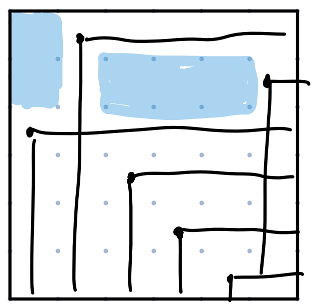
As an example, $w = 2\ 6\ 1\ 3\ 4\ 5$

equiv, can give just $\{2, 6\} \in \binom{[n]}{k}$ as the data

equiv can give a permutation $\lambda \in_k$ $n-k$  $\ell(w) = \#$ boxes, $|\lambda|$

Your subset tells you when to make up steps 

equiv, draw Rothe diagram, shove all boxes «up and to the left» not 100% on this choice
and get a partition in $k \times (n-k)$



We get that $w \le w'$ in Bruhat $\iff \lambda \le \lambda'$

Sources:

[AL 01] Alain Lascoux - Ordering the Affine Symmetric Group

[BB 05] Anders Björner, Francesco Brenti - Combinatorics of Coxeter Groups

[Yas 24] Jad Abou Yassin - Affine Symmetric Groups